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Technical Notes

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In-Plane Warping Effects in Thin-Walled Box Beams

Omri Rand*
Technion—Israel Institute of Technology, 32000 Haifa, Israel

Introduction

THE phenomena related to the ovalization of isotropic long tubes under a pure bending moment have been studied extensively. The pioneer in exploring the nonlinear effects associated with the flexure of long tubes was Brazier,[1] who expressed the flattening of circular cross sections as a single cosine term of the radial component of the in-plane deformation. Further refined models have been reported, for example, in Refs. 2 and 3. Following the aforementioned first study of Brazier, the ovalization of circular tubes is also referred to as Brazier's effect. Detailed reviews of the associate aspects of this problem may be found in Refs. 4–6. In general, it may be stated that the main conclusion that emerges from the preceding studies is that the in-plane deformation in elastic isotropic tubes induces nonlinear bending moment-curvature relationships, which should be accounted for when the bending moment reaches certain levels. Another important conclusion is that there is a maximal value for the moment that may be applied, even when the ultimate stress levels and the associated buckling phenomena are ignored.

Currently, the ability to correctly formulate structural models for thin-walled isotropic and composite beams is important to many general engineering applications, among which helicopter blades and aircraft wings are typical representatives. References 7–11 are representative studies that discuss the influence of warping on the structural analysis of composite beams. Parts of these studies are focused on the coupling effects that are strong functions of the warping modeling. Reference 12 represents a vast range of studies that explored the same phenomena, termed shear deformation in the context of thin laminated plates.

It is common to divide the warping components into two main groups. The first one is the out-of-plane warping that consists of the deformation perpendicular to the cross section, and the second one is the in-plane warping that consists of the deformation in the cross-sectional plane that causes changes in its shape.

As far as the beam behavior is concerned, one is interested mainly in the warping influence on global beam phenomena such as the bending moment-curvature relationship or the bending curvature-twist coupling magnitude (in symmetric beams).

In attempting to support the effort of identifying and establishing the relative importance of the in-plane warping components, an analytic description of the in-plane warping in thin-walled box beams (i.e., rectangular, single-cell cross sections) is offered. The model enables one to predict the effect of the in-plane warping on bending moment-curvature relationships for both isotropic and composite beams under combined loads, and it may be combined with other models that do not include the in-plane deformation, that is, models where the cross sections remain rigid in their own plane.

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*Associate Professor, Faculty of Aerospace Engineering.

Analysis and Discussion

The following discussion will be focused mainly on the effect of a bending moment M_y on a uniform, initially straight isotropic box beam, and later its applicability to the case of torsional moments, composites materials, and general loading will be addressed as well. Figure 1a presents a single-cell, thin-walled rectangular cross section of dimensions a and b and a constant wall thickness t. Based on the experimental results of Ref. 13 and the numerical results of Ref. 14, it is assumed that the in-plane deformation causes the flanges and the webs (i.e., the horizontal and vertical walls, respectively) to deform as curved panels of constant curvature (i.e., to deform into circular panels), as shown in Fig. 1b for the upper-right-hand-side quarter of the cross section. Geometric considerations show that, to maintain a right angle between the flange and the web, the angles created by the flange and the web arcs have to be identical, and they are denoted θ in Fig. 1b. It is further assumed that the cross section deforms inextensionally (similar to Brazier's analysis), and, therefore, the flange and the web lengths should remain constants and, thus, $R_a = a/(2\theta)$ and $R_b = b/(2\theta)$ (which also shows that $R_a/R_b = a/b$). Again, the preceding assumptions regarding the walls deformation as circular panels and the preservation of the angle between the flanges and the webs (excluding small elastic angles) were proved to be valid by the preceding indicated experimental results and detailed numerical models.

Based on the preceding scheme, the deformation may be expressed as a function of a single parameter θ. For a given θ, R_a and R_b are determined as indicated earlier, and the locations y_0 and z_0 (Fig. 1b) are

$$y_0 = R_a \sin(\theta) - R_b \cos(\theta) \tag{1a}$$

$$z_0 = R_b \sin(\theta) + R_a \cos(\theta) \tag{1b}$$

The strain energy components (per unit length) associated with such a deformation are given by

$$U_e^B = (I_{zz}/2)[E/(1-\nu^2)]w,_{xx}^2 \tag{2a}$$

$$U_b = \frac{1}{24}\frac{E}{1-\nu^2}t^3 \oint c^2\,\mathrm{d}s \tag{2b}$$

where U_e^B is the energy associated with the beamwise bending curvature $w,_{xx}$ (w is the beam deformation in the z direction; see Fig. 1a) and U_b is the energy associated with the in-plane bending. In these equations,

$$I_{zz} = \iint z^2\,\mathrm{d}A$$

is the cross-sectional moment of inertia, E is Young's modulus, ν is Poisson's ratio, and c is the wall bending curvature (which is equal to $1/R_a$ for the flanges and $1/R_b$ for the webs). Based on the described deformation, it is clear that

$$\oint c^2\,\mathrm{d}s = \frac{2a}{R_a^2} + \frac{2b}{R_b^2} \tag{3}$$

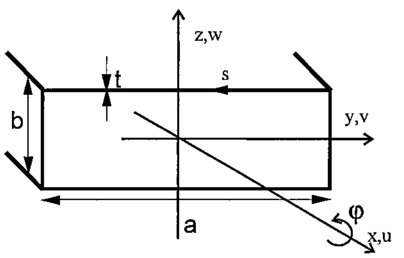


Fig. 1a Box beam notation.

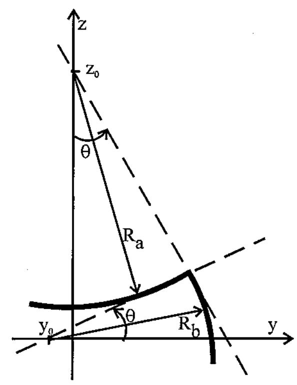


Fig. 1b In-plane warping notation.

Thus, the energy expressions may be written in a nondimensional form as

$$a/t^3[(1-\nu^2)/E]U_e^B = (\bar{I}_{zz}/2)\gamma^2 \tag{4a}$$

$$a/t^3[(1-\nu^2)/E]U_b = \theta^2/3(1+a/b) \tag{4b}$$

where $\bar{I}_{zz} = I_{zz}/(a^3 t)$ and γ is a nondimensional curvature parameter given by

$$\gamma = w_{,xx}(a^2/t) \tag{5}$$

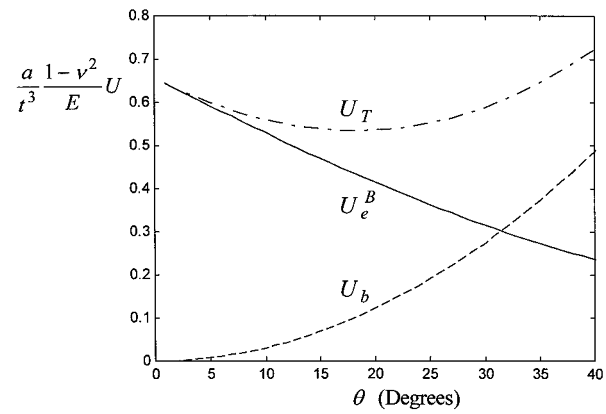
Under the present assumptions, $\bar{I}_{zz}$ is the following function of θ (Fig. 1b):

$$\bar{I}_{zz} = \frac{1}{\theta^2}\left\{\frac{1}{2}\left(\frac{\sin(\theta)}{\alpha} + \cos(\theta)\right)^2 - \frac{\sin^2(\theta)}{\theta\alpha} - \frac{\sin(2\theta)}{8\theta}\left(3 + \frac{1}{\alpha^3}\right) + \frac{1}{4}\left(1 + \frac{1}{\alpha^3}\right)\right\} \tag{6}$$

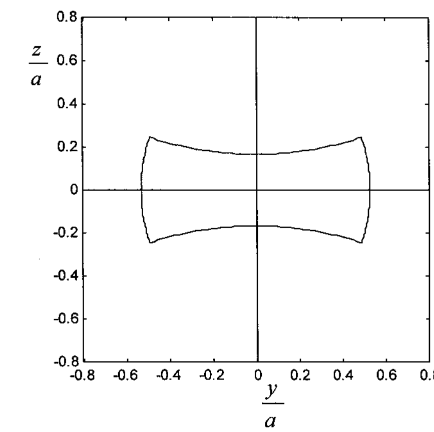
where $\alpha = a/b$. It is easy to show that, for a given beamwise curvature γ, U_e^B is a decreasing function of θ (because I_{zz} is a decreasing function of θ) and U_b is an increasing function of θ. Therefore, the total energy $U_T = U_e + U_b$ may attain a minimum value for a certain value of θ that establishes the cross-sectional deformation at the state of equilibrium.

Figure 2a presents the preceding energy expressions as functions of θ for $\gamma = 3$ and a cross section of $a/b = 2$. As shown, in this case, the total potential energy U_T attains a stationary state for $\theta \cong 18.2$ deg. The resulting cross-sectional deformation for this case is presented in Fig. 2b.

Figure 3a presents the variation of the cross-sectional moment of inertia (normalized by the moment of inertia of the undeformed cross section) obtained by the preceding minimization procedure as a function of the beamwise curvature γ for various values of a/b. In general, it is shown that the cross-sectional moment of inertia is a decreasing function of γ. The decreasing rate of the moment of inertia is smaller for higher a/b ratios. Figure 3b shows the corresponding bending moment M_y as a function of γ. Essentially, this bending moment has been obtained by the engineering



a) Variation of the energy expressions as functions of θ



b) In-plane warping

Fig. 2 Case of $a/b = 2$ and $\gamma = 3$.

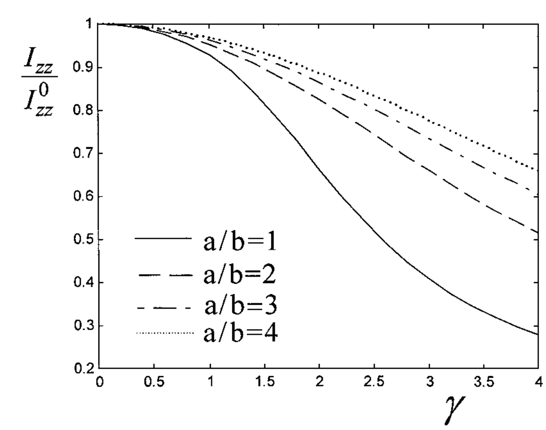


Fig. 3a Cross-sectional moment of inertia as a function of γ for various a/b ratios.

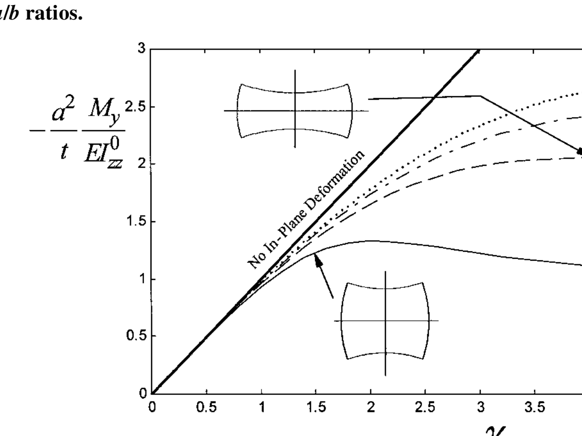


Fig. 3b Bending moment as a function of γ for various a/b ratios, the case of no in-plane deformation, and two examples of the cross-sectional deformation.

bending moment-curvature expression $M_y = -EI_{zz}w_{,xx}$, which becomes

$$-\frac{a^2}{t}\frac{M_y}{EI^0_{zz}} = \frac{I_{zz}}{I^0_{zz}}\gamma \tag{7}$$

When no in-plane warping is included, $I_{zz}/I^0_{zz} = 1$, which yields a linear bending moment-curvature relationship as shown in Fig. 3b. However, when the in-plane warping is included, nonlinear effects should be considered for $\gamma > 1$. Because I_{zz} is a decreasing function of γ (Fig. 3a), the given moment may exhibit a maximum value for certain a/b ratios. Indeed, for $a/b = 1$, Fig. 3b shows that the moment attains a maximum value around $\gamma = 2$, which is similar to the Brazier effect in circular tubes. However, for higher a/b ratios, no maximal value is observed. Two examples for the deformed cross-sectional shape in this case are also presented in Fig. 3b. Note that, in the case of $a/b = 1$, $\gamma = 1.5$ represents almost the maximum possible moment and deformation for this cross section.

When the same cross section undergoes a torsional moment, U^B_e should be replaced by

$$U^T_e = (2tA^2G/p)\phi_{,x}^2 \tag{8}$$

where $\phi_{,x}$ is the twist (about the x axis; see Fig. 1a), A is the area enclosed by the median line, p is the cross-sectional circumference, and G is the shear modulus. Similar to the procedure described for bending, the area A may be easily expressed as a (decreasing) function of θ and the cross-sectional deformation is obtained by the minimization of $U_T = U^T_e + U_b$. Note that, due to the nonlinear nature of the problem, the case of combined bending and torsion moments needs to be determined explicitly by minimizing $U_T = U^B_e + U^T_e + U_b$ and cannot be calculated as a superposition of the bending and the torsion cases.

To determine the in-plane deformation in composite beams, one may adopt the preceding technique and assume that the beamwise bending curvature $w_{,xx}$, the chordwise bending curvature $v_{,xx}$ (v is the beam deformation in the y direction; see Fig. 1a), the axial strain $u_{,x}$ (u is the beam axial deformation in the x direction; see Fig. 1a), the twist $\phi_{,x}$, and the out-of-plane warping function $\Psi(x, s)$ are all known. Thus, the relevant strain components become[15]

$$\varepsilon_{\xi\xi} = u_{,x} - yv_{,xx} - zw_{,xx} + \Psi_{,x} \tag{9a}$$

$$\varepsilon_{\eta\eta} = \zeta c \tag{9b}$$

$$\gamma_{\xi\eta} = -r\phi_{,x} - \Psi_{,s} \tag{9c}$$

where s is a local circumference coordinate that is tangent to the wall direction (see Fig. 1a) and ζ is a local thickness coordinate that is perpendicular to s at each point ($\zeta = 0$ at the wall middle plane). Here r is the normal distance from the tangent to the contour at the point under discussion, for example, for the undeformed cross section, $r = b/2$ over the flanges and $r = a/2$ over the webs. Consequently, the energy terms take the form

$$\bar{U}_e = \frac{1}{2}\int\int\left[Q_{11}(\varepsilon_{\xi\xi})^2 + Q_{66}(\gamma_{\xi\eta})^2 + 2Q_{16}\varepsilon_{\xi\xi}\gamma_{\xi\eta}\right]\mathrm{d}\zeta\,\mathrm{d}s \tag{10a}$$

$$\bar{U}_b = \frac{1}{2}\int\int\left[Q_{22}(\varepsilon_{\eta\eta})^2\right]\mathrm{d}\zeta\,\mathrm{d}s \tag{10b}$$

where Q_{ij} are the elastic moduli that correspond to the material and the ply angle distributions (for a state of plane stress in the present case). Note that the given integrals are performed over the wall thickness, namely, from $\zeta = -t/2$ to $+t/2$ and along the cross-sectional circumference s (see Fig. 1a).

Similar to the isotropic case, the expression for $\bar{U}_e$ is a decreasing function of the deformation angle θ because z and r are both functions of θ, and $\bar{U}_b$ is an increasing function of θ.

An additional convenient way to deal with this system that includes coupled relations between several strain components and several loads is to employ an outer solution scheme that first determines the strains for a given set of loads and for the undeformed cross-sectional geometry. Then the in-plane deformation may be determined as described earlier, and an additional outer solution is executed with the deformed cross-sectional geometry. The preceding two-step iterative procedure is repeated until convergence is achieved (see also Ref. 14). This method provides the in-plane deformation for a given set of loads, as opposed to the method proposed for the isotropic case, where the in-plane deformation for a given value of the beamwise curvature (or the twist) has been obtained. In general, nonlinearities that are similar to those presented in Figs. 3a and 3b have been obtained in Ref. 14 for typical composite beams.

Conclusion

In conclusion, it may be stated that the in-plane warping may play an important role in the structural behavior of thin-walled composite box beams when the nondimensional curvature parameter γ is greater than a unit. Similar to circular tubes, box beams of low a/b ratios have a maximum value for the moment they are capable of supporting, even when ultimate stress levels and buckling phenomena are ignored. Box beams of higher a/b ratios do not exhibit such a maximum value.

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G. A. Kardomateas
Associate Editor